2. **Complex Powers and ‘Heat’ Operators**

Let $P$ be an unbounded operator on a Hilbert space $H$.

**Definition 2.1.** We say that the ray $R_\theta = \{re^{i\theta} \in \mathbb{C} : r \geq 0\}$ is a ray of minimal growth for $P$, provided $R_\theta$ does not intersect the spectrum of $P$ and there exists a constant $c \geq 0$ with

$$\|(P - \lambda)^{-1}\| \leq c\lambda^{-1}.$$  

2.1. **Complex powers.** Complex powers of pseudodifferential operators were first studied in a by now classical paper by Seeley [20]. Let us first recall the definition, which works in a more general context.

Let $R_\theta$ be a ray of minimal growth for the operator $P$. In particular, zero then is not in the spectrum of $P$ and hence there is a $\delta_0 > 0$ such that $B(0, 2\delta_0)$ is contained in the resolvent set. For $\Re s < 0$ define $P_s$ by

$$(2.4) \quad P_s = \frac{i}{2\pi} \int_{\mathcal{C}} \lambda^s(P - \lambda)^{-1} d\lambda.$$ 

Here $\mathcal{C}$ is the contour in $\mathbb{C}$ from $\infty$ to $\delta_0 e^{i\theta}$ along $R_\theta$, clockwise about the circle $\{|z| = \delta_0\}$ to $\delta_0 e^{i\theta}$ and back to $\infty$ along $R_\theta$. The integral converges, since $|\lambda^s| \leq c_\theta|\lambda|^s$.

A crucial point is that on the incoming ray the argument of $\lambda$ is considered to be $\theta$, while on the outgoing ray, it is $\theta - 2\pi$. Hence the pieces along the ray do not cancel unless $s$ is a negative integer.

**Remark 2.2.** Expressions of the form

$$(2.5) \quad f(P) = \frac{i}{2\pi} \int_{\mathcal{C}} f(\lambda)(P - \lambda)^{-1} d\lambda$$

with a contour $\mathcal{C}$ which ‘surrounds’ the spectrum of $P$ and a function $f$ which is holomorphic on the spectrum of $P$ are called Dunford integrals for $f(P)$.

The underlying idea is Cauchy’s theorem in complex analysis: For a holomorphic function on a simply connected domain and a contour $\mathcal{C}$ which simply surrounds $z$

$$f(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(w)}{w - z} dw.$$ 

Note that the shift in the sign is simply due to the fact that we consider $(P - \lambda)^{-1}$ instead of $(\lambda - P)^{-1}$.

The notation $P^s$ is justified by the following theorem:

**Theorem 2.3.** Let $s, t \in \mathbb{C}$ with negative real parts.

(a) $s \mapsto P_s$ is an analytic family of bounded operators

(b) $P_s P_t = P_{s + t}$

(c) $P_{-1} = P^{-1}$ is the inverse to $P$.

**Proof.** (a) follows by differentiating under the integral sign. For (b) let $\mathcal{C}'$ be a contour which lies inside $\mathcal{C}$ and close to $\mathcal{C}$. By Cauchy’s theorem we can then replace $\mathcal{C}$ by $\mathcal{C}'$. Then

$$P_s P_t = -\frac{1}{4\pi^2} \int_{\mathcal{C}'} \left( \int_{\mathcal{C}} (P - \lambda)^{-1}(P - \mu)^{-1} \lambda^s \lambda^t d\lambda \right) d\mu = \frac{1}{4\pi^2} \int_{\mathcal{C}'} \int_{\mathcal{C}} \frac{\lambda^s \lambda^t}{\lambda - \mu} ((P - \lambda)^{-1} - (P - \mu)^{-1}) d\mu d\lambda,$$

$$= \frac{i}{2\pi} \int_{\mathcal{C}'} \lambda^{s+t}(P - \lambda)^{-1} d\lambda + \frac{1}{4\pi^2} \int_{\mathcal{C}'} (P - \lambda)^{-1} \int_{\mathcal{C}'} \frac{\lambda^s \lambda^t}{\lambda - \mu} d\lambda d\mu,$$
where Fubini’s theorem has been applied. The last integral vanishes, since $\mu$ lies outside of $\mathcal{C}'$.

(c) For a negative integer, the integration contour reduces to the circle of radius $\delta_0$ surrounded counterclockwise. Denote by $C$ the opposite contour. We can then write the expression (2.4) for $P^{-1}$ as

$$\frac{1}{2\pi i} \int_C \lambda^{-1} (P - \lambda)^{-1} d\lambda = \frac{1}{2\pi i} \int_C \lambda^{-1} \lambda^{-1} P^{-1} (P^{-1} - \lambda^{-1})^{-1} d\lambda = -\frac{1}{2\pi i} \int_C (P^{-1} - \mu)^{-1} d\mu P^{-1}$$

with the inverse $P^{-1}$ to $P$. Now we observe that the spectrum of $P^{-1}$ lies inside $C$. Holomorphic functional calculus for the bounded operator $P^{-1}$ and $f \equiv 1$ then shows the assertion. □

We can therefore define the powers $P^s$ for all $s \in \mathbb{C}$: We let

$$P^s = \begin{cases} P^s, & \text{Re } s < 0 \\ P^k P^s - k, & k \text{ integer, } -1 \leq \text{Re } s - k < 0. \end{cases}$$

2.2. Heat operators. Instead of only making the assumption of the existence of a ray of minimal growth, we assume that $P - \lambda$ is invertible for all $\lambda$ in a sector

$$\Lambda = \Lambda_\theta = \{ re^{i\varphi} \in \mathbb{C} : r \geq 0 \text{ and } |\varphi| \geq \theta \}$$

for some $\theta < \pi/2$ (2.6) and that

$$\| (P - \lambda)^{-1} \| \leq c |\lambda|^{-1}, \quad \lambda \in \Lambda,$$

for a suitable constant $c$.

For $t > 0$ we then define

$$e^{-tP} = \frac{i}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} (P - \lambda)^{-1} d\lambda,$$

where $\mathcal{C}$ is the contour from $\infty$ to $\delta_0 e^{i\theta}$ along the ray $R_\theta$, clockwise about the origin on the circle $|z| = \delta_0$ to $\delta_0 e^{-i\theta}$ and back to $\infty$ along $R_{-\theta}$.

The integral converges, since $e^{-tM}$ decays exponentially along the rays. Note that for this to be the case it is important that the rays lie in the right half plane.

Remark 2.4. The name ‘heat’ operator stems from the fact that $e^{-tP} u_0$ solves the equation $\partial_t u + P u = 0$, $u(0) = u_0$, which becomes the heat equation for $P = -\Delta$.

Theorem 2.5. (a) $t \mapsto e^{-tP}$ is a smooth function on $R_{>0}$ with values in bounded operators.

(b) Let $s, t > 0$. Then $e^{-sP} e^{-tP} = e^{-(s+t)P}$.

Proof. (a) follows by differentiating under the integral sign. For (b) use a similar argument as in the proof of Theorem 2.3 □

2.3. Domains. In the above, we are making assumptions on the spectrum of the operator $P$ as an unbounded operator on a Hilbert space. As a consequence we have to specify the domain of $P$. In general, there are many possible choices. We will focus, however, on the case where $P$ is an elliptic pseudodifferential operator (in fact, we will make even stronger assumptions on $P$) considered as an unbounded operator on $L^2(X)$. In that case, there only is one closed extension.
Definition 2.6. Let \( A : C^\infty(X) \to C^\infty(X) \) be an arbitrary operator. By \( D_{\text{min}} \), the minimal domain, we denote the domain of the closure of \( A \), while \( D_{\text{max}} \) is the set of all \( u \in L^2 \) such that \( Au \in L^2 \).

Clearly, \( D_{\text{min}} \) is the domain of the smallest closed extension and \( D_{\text{max}} \) that of the largest.

Theorem 2.7. Let \( P \) be an elliptic pseudodifferential operator of order \( \mu > 0 \). Then \( D_{\text{min}} = D_{\text{max}} = H^\mu(X) \).

Proof. Let \( u \in H^\mu(X) \). Then there exists a sequence \( u_m \in C^\infty(X) \) with \( u_m \to u \in H^\mu(X) \). Hence \( H^\mu \subseteq D_{\text{min}} \). Conversely, suppose that \( u \in D_{\text{max}} \), i.e. \( u \in L^2 \) and \( Pu \in L^2 \). Elliptic regularity then implies that \( u \in H^\mu(X) \). Hence \( D_{\text{max}} \subseteq H^\mu \). This shows the assertion. \( \square \)

Theorem 2.8. Let \( P \) be an elliptic pseudodifferential operator of order \( \mu > 0 \). Then either the \( L^2 \)-spectrum of \( P \) is all of \( \mathbb{C} \), or it consists of a countable number of eigenvalues with no accumulation point.

Proof. If \( P - \lambda \) is invertible for some \( \lambda \), then

\[
(P - \lambda)^{-1} : L^2 \to \mathcal{D}(P) = H^\mu \hookrightarrow L^2
\]

is compact. This implies that the spectrum of \( (P - \lambda)^{-1} \) is discrete with only possible accumulation point in zero. This shows the assertion since the spectral values of \( P - \lambda \) are just the inverses of the elements in the spectrum of \( (P - \lambda)^{-1} \). \( \square \)

2.4. Strategy. In a first step, we shall see that the resolvent can be replaced by a parameter-dependent parametrix with a classical symbol having special homogeneity properties. This is the decisive step for the construction of both \( P^s \) and \( e^{-tP} \).

We shall see that \( P^s \) is a pseudodifferential operator of order \( \mu \text{ Re } s \), where \( \mu \) is the order of \( P \) and that \( e^{-tP} \) is a smoothing operator. According to Theorem 1.16 it makes sense to take the trace of \( e^{-tP} \) and that of \( A^s \), provided \( \text{Re } s < -n/\mu \).

From the asymptotic expansion of the parametrix symbol we then derive (under suitable additional assumptions on \( P \) and its symbol), the meromorphic structure of the trace of \( P^s \) and the asymptotic expansion of the trace of \( e^{-tP} \).

2.5. Notes. The fundamental paper here is Seeley’s article [20] on complex powers, where also the strategy of the resolvent analysis was developed. Kumano-go and Tsutusmi [15] simplified the technique; it is worthwhile having a look at Kumano-go’s book [14]. In principle, the same information can be extracted from the traces of the resolvent, the heat kernel, and the complex powers, see [9]. For pseudodifferential boundary value problems, corresponding results on asymptotic expansions are harder to obtain, see Grubb’s book [6].

In connection with noncommutative residues, for example, it is important to consider not only the traces of \( P^s \) or \( e^{-tP} \), but more generally traces of operators \( QP^s \) or \( Qe^{-t} \) for general pseudodifferential operators \( Q \). In the closed manifold case, the analysis is mostly parallel. For boundary value problems see e.g. [7].